ROBUST DICKEY-FULLER TESTS BASED ON RANKS FOR TIME SERIES WITH ADDITIVE OUTLIERS

V. A. REISEN, C. LÉVY-LEDUC, M. BOURGUIGNON, AND H. BOISTARD

Abstract. In this paper the unit root tests proposed by Dickey and Fuller (DF) and their rank counterpart suggested by Breitung and Gouriéroux (1997) (BG) are analytically investigated under the presence of additive outlier (AO) contaminations. The results show that the limiting distribution of the former test is outlier dependent, while the latter one is outlier free. The finite sample size properties of these tests are also investigated under different scenarios of testing contaminated unit root processes. In the empirical study, the alternative DF rank test suggested in Granger and Hallman (1991) (GH) is also considered. In Fotopoulos and Ahn (2003), these unit root rank tests were analytically and empirically investigated and compared to the DF test, but with outlier-free processes. Thus, the results provided in this paper complement the studies of the previous works, but in the context of time series with additive outliers. Equivalently to DF and Granger and Hallman (1991) unit root tests, the BG test shows to be sensitive to AO contaminations, but with less severity. In practical situations where there would be a suspicion of additive outlier, the general conclusion is that the DF and Granger and Hallman (1991) unit root tests should be avoided, however, the BG approach can still be used.

Keywords: Time Series; robust tests; ranks; outliers; unit root.

1. Introduction

An important issue in time series analysis is the determination of the degree of integration. In the standard ARIMA(p, d, q) model, the degree d of integration is defined as the number of times the series must be differenced to yield a stationary time series with invertible MA representation (Breitung (1994)). Then, the order of integration of a time series is a crucial determinant of the properties exhibited by the series and, therefore, unit root tests play an important role in the analysis of non-stationary time series. Since the seminal works by Dickey and Fuller (1979, 1981) and Said and Dickey (1984), usually referred as DF and ADF tests respectively, testing unit root has become routine in empirical studies, specially in the economic area.

Since the early 90's, various generalizations of DF and ADF tests have been proposed for handling unit root processes, see for example, Elliot et al. (1996), Aparicio et al. (2006), Demetrescu et al. (2008) and Astill et al. (2013) and Westerlund (2014) for a recent review. Among all the research dedicated to unit root processes, a special attention has been paid to

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testing unit root against fractional alternatives, and the studies have indicated that the power of DF test is low in such circumstances (see Hassler and Wolters (1994); Kramer (1998); Wang (2011) and references therein). To improve the power and the size of unit root tests, bootstrap techniques have been considered in a large number of papers as an alternative way to distinguish long-memory processes from more common I(0) and I(1) processes, see for example, Franco and Reisen (2007) and Palm et al. (2008). Demetrescu et al. (2008) proposed an integration test against fractional alternatives which is based on the pioneering Lagrange multiplier (LM) test suggested by Robinson (1991) and further studied by Breitung and Hassler (2002) among others.

An alternative nonparametric unit root test, the DF test based on ranks (DF-rank test), to deal with non-linear transformations was firstly discussed in Granger and Hallman (1991), further studied by Breitung and Gouriéroux (1997) and by Fotopoulos and Ahn (2003). The DF-rank tests do not use the observations but rather the ranks of the observations which makes the tests relatively insensitive to outlying observations. A discussion of the advantage of the nonparametric unit root test over the parametric one is given in the introduction section of Breitung and Gouriéroux (1997).

Under some assumptions, Breitung and Gouriéroux (1997) derived the asymptotic behavior of their test under the null hypothesis. In the same direction, Fotopoulos and Ahn (2003) considered comparisons between Breitung and Gouriéroux (1997) and Granger and Hallman (1991) rank statistics for the unit root test. The authors also provide the limiting distribution of the Breitung and Gouriéroux (1997) test under the null hypothesis and give small sample properties. In these works, the tests were built and empirically tested under the alternative of AR processes. The empirical study undertaken by Fotopoulos and Ahn (2003) indicated that the DF test showed higher power than the rank test counterparts. An alternative robust approach based on ranks to test unit root was recently proposed by Hallin et al. (2011), and they showed that their method is robust with respect to the innovation density.

An advantage of the DF-rank test over the parametric is that the nonparametric approach is, in general, outlier-resistant. As is well-known, the outliers may seriously destroy the statistical property of an estimator (Chan (1995)). See, for example, Martin and Yohai (1986) in the case of the least square estimator and the recent works by Molinares et al. (2009) and by Lévy-Leduc et al. (2011b,a) and Reisen and Molinares (2012) in the context of time series with short and long-memory properties. Franses and Haldrup (1994) investigated the size of the DF test in the presence of additive outliers. These authors provide analytical as well as empirical evidences that AO may produce spurious stationarity, that is, the additive outliers provoke the rejection of unit root too often. The recent work Astill et al. (2013) suggests a bootstrap test for detecting additive AO in a univariate unit root process. As an alternative way to testing unit root with data anomalies and, also, with different error distributions, M-estimators have been invoked in some studies, for example, Lucas (1995) and Carstensen (2003).

Although the M-robust unit root tests are in some way better protected against some deviations from gaussianity, however, under additive outliers they fail to produce stable sizes and good power properties. Their size distortions are similar to the DF test.

The work reported here complements the research of Granger and Hallman (1991), Breitung and Gouriéroux (1997), Fotopoulos and Ahn (2003) and Hallin et al. (2011) by providing the asymptotic distribution of the rank tests for unit root processes with additive outliers. Under outliers contaminations, the size and the power of the tests are investigated empirically for finite sample sizes. The rest of this paper is organized as follows. Section 2 discusses the analytical results of the unit root testings whether the process is contaminated with additive outliers or not. In Section 3, the tests are investigated for small sample sizes in different scenarios of the unit root processes. Conclusions are given in Section 4.

2. Theoretical results

In this section some theoretical results are derived and discussed. Let

$$y_t = \rho y_{t-1} + \varepsilon_t \ , \ t = 1, \ldots, T \ , \tag{1}$$

where $|\rho| \leq 1$, $y_0 = 0$ and ε_t's are i.i.d. random variables with mean zero and variance σ^2. If $\rho = 1$, the process (1) is nonstationary in the mean and it is known as random walk or integrated process of order 1 (I(1)). On the other hand, the bound $|\rho| < 1$ implies stability and asymptotic stationarity properties of the process y_t.

The type of outliers has quite different impact and properties for models and robust estimates. In general, the innovation (IO) and additive outliers are the most common type of atypical observations discussed in the literature (see, for example, Chen and Liu (1993)). Under the assumption of IO, the least squares estimates of ρ are consistent even when the error in the model is not Gaussian but with a finite variance. As is well-known, AO are known to deteriorate more the estimator than IO. In the presence of AO, the least squares estimates not only lack robustness in terms of variability but also suffer from bias problems (see, for example, Denby and Martin (1979) and Franses and Haldrup (1994)). In the case of unit root model (I(1)), the AO provokes a spurious model estimation by indicating that the process has stationary roots.

The model contaminated by additive outliers is defined here as

$$z_t = y_t + \theta\delta_t \ , \ t = 1, \ldots, T \ , \tag{2}$$

where θ is the magnitude of the outliers (fixed and unknown parameter), δ_t's are i.i.d. random variables, with variance σ_δ^2, such that $\mathbb{P}(\delta_t = 1) = \mathbb{P}(\delta_t = -1) = \pi/2$ and $\mathbb{P}(\delta_t = 0) = 1 - \pi$, where π is in $(0, 1)$. It is assumed here that the random variables δ_t's are independent of y_t's. The terms $\theta\delta_t$ in (2) are called effects rather than errors since it causes an immediate and one-shot effect on the observed series. As is discussed in the literature, the AO seems to be a more appropriate descriptor of non repetitive events (see, for example, Denby and Martin (1979) and Franses and Haldrup (1994)).

In the sequel, the asymptotic behavior of two different unit-root tests under the null hypothesis "$\rho = 1$" is discussed when the process is contaminated with AO.

2.1. **Dickey-Fuller test under outliers.** Following the regression defined in (2), the least square estimates is

$$\hat{\rho}_{\mathrm{DF}} - 1 = \frac{\sum_{t=1}^{T} z_{t-1}\Delta z_t}{\sum_{t=1}^{T} z_{t-1}^2} \,, \tag{3}$$

where $\Delta z_t = z_t - z_{t-1}$. Under the null hypothesis (H_0): "$\rho = 1$", the DF test statistics and their limiting distributions are given as follows

$$T(\hat{\rho}_{\mathrm{DF}} - 1) = T\frac{\sum_{t=1}^{T} z_{t-1}\Delta z_t}{\sum_{t=1}^{T} z_{t-1}^2} \,, \tag{4}$$

and

$$\hat{\tau}_{\mathrm{DF}} = \frac{\hat{\rho}_{\mathrm{DF}} - 1}{\hat{\sigma}_{\mathrm{DF}} \left(\sum_{t=1}^{T} z_{t-1}^2\right)^{-1/2}} \,, \tag{5}$$

where $\hat{\sigma}^2_{\mathrm{DF}} = T^{-1}\sum_{t=1}^{T}(\Delta z_t - (\hat{\rho}_{\mathrm{DF}} - 1)z_{t-1})^2$.

Theorem 1. *Assume that $(z_t)_{1\leq t\leq T}$ are given by (2), then when $\rho = 1$,*

$$T(\hat{\rho}_{DF} - 1) \xrightarrow{d} \frac{W(1)^2 - 1}{2\int_0^1 W(r)^2 \mathrm{d}r} - \left(\frac{\theta}{\sigma}\right)^2 \frac{\pi}{\int_0^1 W(r)^2 \mathrm{d}r} \,, \quad \text{as } T \to \infty \,, \tag{6}$$

$$\hat{\tau}_{DF} \xrightarrow{d} [1 + 2(\theta/\sigma)^2\pi]^{-1/2} \left\{ \frac{W(1)^2 - 1}{2\left(\int_0^1 W(r)^2 \mathrm{d}r\right)^{1/2}} - \frac{(\theta/\sigma)^2\,\pi}{\left(\int_0^1 W(r)^2 \mathrm{d}r\right)^{1/2}} \right\} \,, \quad \text{as } T \to \infty \,, \tag{7}$$

where $\xrightarrow{d}$ denotes the convergence in distribution and where $\{W(r),\ r \in [0,1]\}$ denotes the standard Brownian motion.

The proof of Theorem 1 is given in Franses and Haldrup (1994). We have also written the proof with full details and it is available upon request. The standard DF tests are a particular case of (6) and (7) when either $\theta = 0$ or $\pi = 0$. As argued by Franses and Haldrup (1994) and also discussed here, for $\theta \neq 0$, the limiting distributions of the tests depend on the parameter θ and, therefore, the behavior of the test distributions will be shifted to the left side according to the size of θ and, also, to the probability of the occurrence of outliers. This translation will inflate the size of the test by leading to reject (H_0) too often in favor of a spurious stationary solution. Based on these properties, the DF test under a suspicion of outliers in the data should be avoided (see Franses and Haldrup (1994)).

2.2. **Breitung-Gouriéroux test.** The Breitung-Gouriéroux's statistics (Breitung and Gouriéroux (1997)) consists in replacing the first difference of the observations $(\Delta y_t)_{1\leq t\leq T}$ by their ranks $(r_{T,t})$ in the Dickey-Fuller test statistics. Then, by using the same computations as those used in Fotopoulos and Ahn (2003), the unit root tests can be written as follows

$$T(\hat{\rho}_{\mathrm{BG}}-1) = T\frac{\sum_{t=2}^{T}\left(r_{T,t}-\frac{T+1}{2}\right)\left\{\sum_{j=1}^{t-1}\left(r_{T,j}-\frac{T+1}{2}\right)\right\}}{\sum_{t=2}^{T}\left\{\sum_{j=1}^{t-1}\left(r_{T,j}-\frac{T+1}{2}\right)\right\}^2}\ , \tag{8}$$

$$\hat{\tau}_{\mathrm{BG}} = \frac{\hat{\rho}_{\mathrm{BG}}-1}{\hat{\sigma}_{\mathrm{BG}}}\left[\sum_{t=2}^{T}\left\{\sum_{j=1}^{t-1}\left(r_{T,j}-\frac{T+1}{2}\right)\right\}^2\right]^{1/2}\ , \tag{9}$$

where

$$\hat{\sigma}^2_{\mathrm{BG}} = \frac{1}{T-2}\sum_{t=2}^{T}\left\{\sum_{j=1}^{t}\left(r_{T,j}-\frac{T+1}{2}\right)-\hat{\rho}_{\mathrm{BG}}\sum_{j=1}^{t-1}\left(r_{T,j}-\frac{T+1}{2}\right)\right\}^2\ . \tag{10}$$

Contrary to the limiting distributions of DF tests, it is proved in the following theorem that the limiting distribution of BG test statistics are outliers-free.

Theorem 2. *Assume that $(z_t)_{1\leq t\leq T}$ are given by (2), then when $\rho=1$,*

$$T(\hat{\rho}_{BG}-1)\xrightarrow{d}-\frac{1}{2\int_0^1 B(r)^2\mathrm{d}r}\ ,\quad \text{as } T\to\infty\ , \tag{11}$$

$$\hat{\tau}_{BG}\xrightarrow{d}-\frac{1}{2\left\{\int_0^1 B(r)^2\mathrm{d}r\right\}^{1/2}}\ ,\quad \text{as } T\to\infty\ , \tag{12}$$

where $\{B(r),\ r\in[0,1]\}$ denotes the standard Brownian bridge, $\hat{\rho}_{BG}$ and $\hat{\tau}_{BG}$ are defined in (8) and (9), respectively, $r_{T,t}$ denoting the rank of $(\Delta z_t)_{1\leq t\leq T}$.

The proof of Theorem 2 is given in Section 5. From Theorem 2, it can be seen that the statistic (11) has the same order as in (6). It can also be observed that the limiting distributions of the BG statistics are always negative. This implies that the estimate of ρ is asymptotically smaller than 1 and this particular problem was seen with some negative criticism by Fotopoulos and Ahn (2003). As an alternative form to overcome this, the authors investigated and suggested the statistics given by Granger and Hallman (1991)(GH). However, as will be seen in the simulation section 3, the empirical distribution of this statistic carries the same problem as the DF tests when the data has outliers, that is, the empirical results indicate that its limiting distribution also depends on the parameter θ. This problem motivated the dedication of this paper to the study of the theoretical properties of the BG statistics under the presence of additive outliers.

3. Small-sample properties

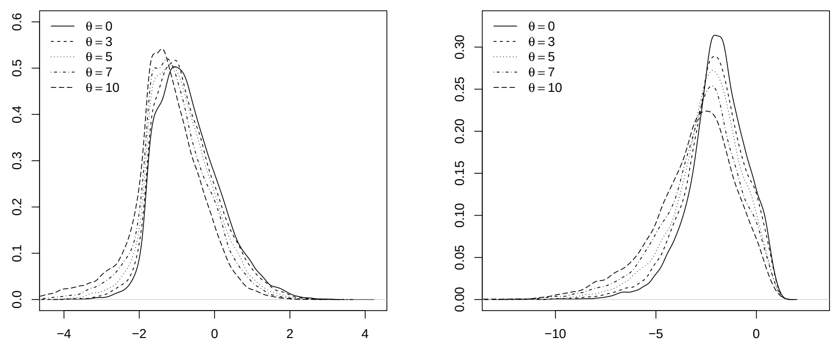
The BG test along with the standard DF and GH tests are applied to unit root processes under the presence of AO with the aim to analyze their performances for finite sample sizes. This empirical investigation will give a better understanding of whether or not the outliers can affect the size of the tests and if it does, how.

For this objective, samples following Equations 1 and 2, where the ε_t's are i.i.d. Gaussian random variables with unit variance, were simulated according to the following scenarios: time series with small and large sample sizes and different outlier magnitudes with fixed probability occurrences of π. The empirical rejection rates of the null hypothesis of unit root are based on 10,000 replications. From now on, the nominal level of the tests is 5%. Table 1 displays the size of the tests without ($\theta = 0$) and under ($\theta \neq 0$) outliers in the data with $\pi = 0.01$. The results for $\pi = 0.05$ are available upon request. The sizes of the tests are also investigated under a fixed number of outliers $T\pi = 1$ and $n = 50, 400$ (the results are in Table 2). Other cases such as $T\pi = 2$ and $n = 100, 200, 1000$ displayed similar conclusions and are available upon request.

Table 1. Rejection rates of $H_0 : \rho = 1$, with $\pi = 0.01$

sample sizes	Tests	Empirical critical points	$\theta = 0$	$\theta = 3$	$\theta = 5$	$\theta = 7$	$\theta = 10$
$T = 50$	$\hat{\tau}_{\mathrm{DF}}$	-1.95	0.0462	0.0723	0.1067	0.1550	0.2125
	$T(\hat{\rho}_{\mathrm{DF}} - 1)$	-7.70	0.0443	0.0707	0.1078	0.1563	0.2149
	$\hat{\tau}_{\mathrm{GH}}$	-1.75	0.0468	0.0449	0.0504	0.0597	0.0546
	$T(\hat{\rho}_{\mathrm{GH}} - 1)$	-3.82	0.0550	0.0733	0.0932	0.1073	0.1088
	$\hat{\tau}_{\mathrm{BG}}$	-2.66	0.0492	0.0636	0.0585	0.0564	0.0602
	$T(\hat{\rho}_{\mathrm{BG}} - 1)$	-12.38	0.0517	0.0602	0.0614	0.0601	0.0641
$T = 100$	$\hat{\tau}_{\mathrm{DF}}$	-1.95	0.0494	0.0709	0.1185	0.1705	0.2549
	$T(\hat{\rho}_{\mathrm{DF}} - 1)$	-7.90	0.0492	0.0711	0.1183	0.1723	0.2591
	$\hat{\tau}_{\mathrm{GH}}$	-1.76	0.0497	0.0570	0.0685	0.0758	0.0880
	$T(\hat{\rho}_{\mathrm{GH}} - 1)$	-4.20	0.0548	0.0774	0.1021	0.1355	0.1569
	$\hat{\tau}_{\mathrm{BG}}$	-2.64	0.0519	0.0533	0.0619	0.0636	0.0604
	$T(\hat{\rho}_{\mathrm{BG}} - 1)$	-13.04	0.0541	0.0568	0.0635	0.0650	0.0620
$T = 200$	$\hat{\tau}_{\mathrm{DF}}$	-1.95	0.0453	0.0741	0.1080	0.1752	0.2729
	$T(\hat{\rho}_{\mathrm{DF}} - 1)$	-7.90	0.0462	0.0754	0.1117	0.1819	0.2835
	$\hat{\tau}_{\mathrm{GH}}$	-1.77	0.0460	0.0603	0.0740	0.1042	0.1282
	$T(\hat{\rho}_{\mathrm{GH}} - 1)$	-4.49	0.0493	0.0811	0.1148	0.1592	0.2161
	$\hat{\tau}_{\mathrm{BG}}$	-2.63	0.0462	0.0587	0.0563	0.0594	0.0612
	$T(\hat{\rho}_{\mathrm{BG}} - 1)$	-13.31	0.0481	0.0598	0.0577	0.0614	0.0638
$T = 400$	$\hat{\tau}_{\mathrm{DF}}$	-1.95	0.0508	0.0741	0.1132	0.1749	0.2958
	$T(\hat{\rho}_{\mathrm{DF}} - 1)$	-8.00	0.0535	0.0775	0.1195	0.1840	0.3080
	$\hat{\tau}_{\mathrm{GH}}$	-1.77	0.0484	0.0577	0.0928	0.1281	0.1953
	$T(\hat{\rho}_{\mathrm{GH}} - 1)$	-4.67	0.0517	0.0754	0.1268	0.1932	0.2900
	$\hat{\tau}_{\mathrm{BG}}$	-2.61	0.0500	0.0637	0.0695	0.0622	0.0639
	$T(\hat{\rho}_{\mathrm{BG}} - 1)$	-13.40	0.0500	0.0643	0.0698	0.0623	0.0646
$T = 1000$	$\hat{\tau}_{\mathrm{DF}}$	-1.95	0.0463	0.0751	0.1202	0.1746	0.2930
	$T(\hat{\rho}_{\mathrm{DF}} - 1)$	-8.00	0.0476	0.0771	0.1220	0.1800	0.3044
	$\hat{\tau}_{\mathrm{GH}}$	-1.77	0.0507	0.0701	0.1038	0.1663	0.2749
	$T(\hat{\rho}_{\mathrm{GH}} - 1)$	-4.86	0.0483	0.0816	0.1407	0.2279	0.3664
	$\hat{\tau}_{\mathrm{BG}}$	-2.62	0.0481	0.0618	0.0597	0.0600	0.0582
	$T(\hat{\rho}_{\mathrm{BG}} - 1)$	-13.64	0.0481	0.0619	0.0598	0.0599	0.0584

It is well-known that the Ordinary Least Square estimator is sensitive to the occurrence of outliers in the data. This sensitivity is inherited by the DF-test as is observed in the table. The empirical size of DF test is inflated due to the sizes of θ and T, and this leads to reject a unit root too often indicating a spurious stationary. This confirms the analytical results given in Theorem 1 and also in Franses and Haldrup (1994). In contrast to the DF test, the rank tests are much less sensitive to additive outliers. Regardless of the sample size and magnitude of the outliers, the results in Table 1 show that the sizes of the tests based on the BG method do not fluctuate too much. They are always near to 5%. However, the GH approach does not always guarantee this empirical property. This test is also sensitive to outliers, but with less impact from the effect of the magnitude and the sample size compared with DF test.

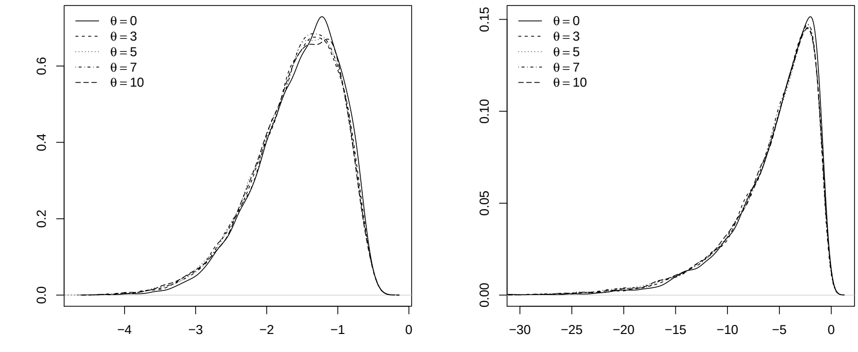


(a) $\hat{\tau}_{\text{GH}}$ (b) $T(\hat{\rho}_{\text{GH}} - 1)$

FIGURE 1. Empirical densities of $\hat{\tau}_{\text{GH}}$ and $T(\hat{\rho}_{\text{GH}} - 1)$ for different values of θ, $T = 200$ and $\pi = 0.01$.

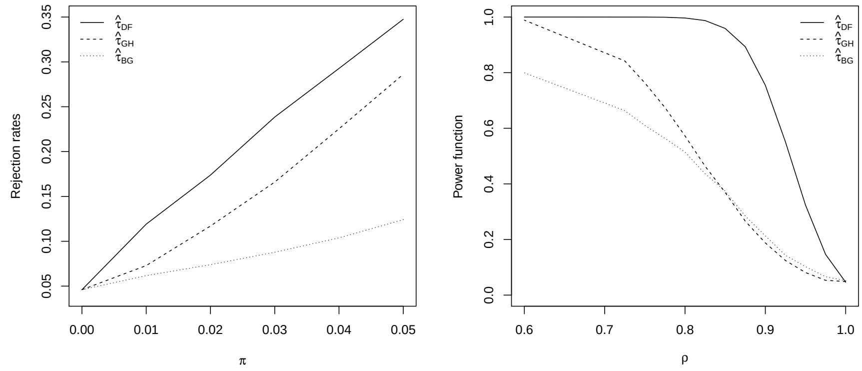
Figures 1 and 2 give the empirical densities of the GH and BG tests, respectively, for $T = 200$. The density of GH test shifts to the left with respect to the distribution of uncontaminated data according to the sizes of θ. The density of DF displays similar behaviour and it is available upon request. On the other hand, the empirical distributions of BG shows to be very similar regardless of the size of θ which confirm the results in Table 1 that this test has an approximate constant size under a variety of contaminations.

As previously stated, Theorem 2 shows that the asymptotical distributions of BG tests do not depend on the size of θ. However, this outlier-free property does not always hold for finite sample sizes. As an example of the size distortions of the tests, these quantities are plotted in Figure 3 (a) for T and θ fixed and π varying from 0(0.01) to 0.05. It turns out that BG test also shows to have the same problems as DF and GH tests, but it is less severe. It is obvious that the sizes of DF and GH tests deteriorate even more with growing θ and T, which were expected results.



(a) $\hat{\tau}_{\mathrm{BG}}$ (b) $T(\hat{\rho}_{\mathrm{BG}} - 1)$

FIGURE 2. Empirical densities of $\hat{\tau}_{\mathrm{BG}}$ and $T(\hat{\rho}_{\mathrm{BG}} - 1)$ for different values of θ, $T = 200$ and $\pi = 0.01$.



(a) (b)

FIGURE 3. (a) Size distortions of the tests for different π, $T = 200$ and $\theta = 5$; (b) Empirical size-adjusted powers, $T = 100$, $\theta = 3$ and $\pi = 0.01$

Table 2 displays the sizes of the tests for fixed outlier numbers $T\pi = 1$, $n = 50, 400$. The effects on the size of the tests do not become serious by the increasing of the sample size, that is, size distortions are less pronounced for all tests. Results for $n = 100, 200, 1000$ presented similar conclusions and are available upon request.

Based on the analytical and empirical properties of the tests discussed here, a general conclusion is that the DF test under a suspicion of outliers in the data should be avoided, see also Franses and Haldrup (1994). The rank test based on GH is somewhat sensitive to

TABLE 2. Rejection rates of $H_0 : \rho = 1$, setting $T\pi = 1$

sample sizes	$\theta = 0$	$\theta = 3$	$\theta = 5$	$\theta = 7$	$\theta = 10$
$T = 50$	0.0497	0.0946	0.1650	0.2436	0.3475
	0.0463	0.0935	0.1681	0.2440	0.3536
	0.0508	0.0586	0.0704	0.0798	0.0881
	0.0539	0.0979	0.1374	0.2440	0.1847
	0.0522	0.0680	0.0728	0.0720	0.0729
	0.0547	0.0716	0.0761	0.0754	0.0766
$T = 400$	0.0474	0.0533	0.0648	0.0873	0.1156
	0.0480	0.0544	0.0666	0.0867	0.1185
	0.0514	0.0488	0.0538	0.0663	0.0776
	0.0532	0.0570	0.0661	0.0821	0.1067
	0.0493	0.0553	0.0536	0.0597	0.0548
	0.0494	0.0555	0.0540	0.0597	0.0551

outliers as the DF test is. Hence, caution also has to be taken when using this test under data contaminations for small sample sizes. Although the BG test also presented some size distortions this test seems to be an alternative unit root test to be used in most of scenarios discussed in this paper.

The empirical powers of the tests were also considered in the scenario of unit root against the alternative of stationary AR processes, but with outlier contaminations. Since the unit root tests presented size distortions under outliers in the data, their empirical powers properties were obtained under size-adjusted power which is not a realistic power in practical situations (see also Lucas (1995), page 165). Figure 3 (b) gives some insights of the lower size-adjusted power of the rank tests against DF test. Other cases are available upon request. In all case considered, the results were not surprising by showing that the powers of the rank tests were inferior to the DF test which are somewhat in accordance with the one discussed in Fotopoulos and Ahn (2003), for no outlier contamination cases (see also (Granger and Hallman, 1991, p. 217)).

4. Conclusions

In this paper robust unit root tests based on ranks, initially suggested in Granger & Hallman (1991) and considered in Fotopoulos & Ahn (2003), are studied theoretically and empirically underlying additive outliers in the level of non stationary time series by complementing the findings of the previous works. In this context, the analytical results show that the limiting distribution of the Dickey-Fuller rank test based on the ranks of ΔY_t is outlier free in contrast with the standard DF test which depends on the magnitude of outliers. The limiting distribution of this test will be shifted to the left and, in consequence, the null hypothesis of unit root will be rejected too often. These theoretical findings were closely mimicked by the Monte Carlo experiments. Samples from non stationary processes were generated under different scenarios of outlier contaminations. In the simulation study, the Granger and Hallman (1991) test was also considered and, equivalently to DF unit root tests, it showed to be sensitive to

AO contaminations, but with less severity. The general conclusion is that, in practical situations where there would be a suspicion of additive outlier, the DF and Granger and Hallman (1991) unit root tests should be avoided, however, the BG approach can still be used.

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5. Proofs

5.1. **Proof of Theorem 2.** Let us start by proving (11). Let us first prove that the numerator of (8) is not random but is just a fixed function of T.

$$\sum_{t=2}^{T}\left\{\left(r_{T,t}-\frac{T+1}{2}\right)\sum_{j=1}^{t-1}\left(r_{T,j}-\frac{T+1}{2}\right)\right\}=\sum_{1\leq j<t\leq T}\left(r_{T,t}-\frac{T+1}{2}\right)\left(r_{T,j}-\frac{T+1}{2}\right)$$
$$=\frac{\left[\sum_{j=1}^{T}\left(r_{T,j}-\frac{T+1}{2}\right)\right]^2-\sum_{j=1}^{T}\left(r_{T,j}-\frac{T+1}{2}\right)^2}{2}$$
$$=-\frac{1}{2}\sum_{j=1}^{T}r_{T,j}^2+\frac{T+1}{2}\left(\sum_{j=1}^{T}r_{T,j}\right)-\frac{T}{2}\left(\frac{T+1}{2}\right)^2=-\frac{T(T+1)(T-1)}{24}\ , \quad (13)$$

where we used that $\sum_{j=1}^{T}(r_{T,j}-(T+1)/2)=0$ and that $\sum_{j=1}^{T}r_{T,j}^2=T(T+1)(2T+1)/6$.

By (13), studying the asymptotic behavior of (11) amounts to studying the following quantity

$$\frac{1}{T^4}\sum_{t=2}^{T}\left\{\sum_{j=1}^{t-1}\left(r_{T,j}-\frac{T+1}{2}\right)\right\}^2=\frac{1}{T^4}\sum_{t=2}^{T}\left\{\sum_{j=1}^{t-1}\left[\left(\sum_{k=1}^{T}\mathbf{1}_{\{\Delta z_k\leq\Delta z_j\}}\right)-\frac{T+1}{2}\right]\right\}^2$$
$$=\frac{1}{T}\sum_{t=2}^{T}\left\{\frac{1}{\sqrt{T}}\sum_{j=1}^{t-1}\left(F_T(\zeta_j)-\frac{T+1}{2T}\right)\right\}^2\ , \quad (14)$$

where $F_T(x)=T^{-1}\sum_{t=1}^{T}\mathbf{1}_{\{\zeta_t\leq x\}}$, where

$$\zeta_t=\Delta z_t=\varepsilon_t+\theta(\delta_t-\delta_{t-1})\ . \quad (15)$$

Let us now prove that

$$\left\{\frac{1}{\sqrt{T}}\sum_{i=1}^{[Tu]}\left(F_T(\zeta_i)-\frac{T+1}{2T}\right)\ ,\ u\in[0,1]\right\}\overset{d}{\Longrightarrow}\{B(u),\ u\in[0,1]\}\ ,\ \text{as } T\to\infty\ , \quad (16)$$

where $[x]$ denotes the integer part of x, $\{B(u),\, u\in[0,1]\}$ denotes the Brownian bridge and $\overset{d}{\Longrightarrow}$ denotes here the weak convergence in the space of cadlag functions of $[0,1]$ denoted $\mathcal{D}([0,1])$ and equipped with the topology of uniform convergence. Splitting $\sum_{j=1}^{T}$ into $\sum_{j=1}^{[Tu]}$ and $\sum_{j=[Tu]+1}^{T}$, we get

$$\frac{1}{\sqrt{T}}\sum_{i=1}^{[Tu]}\left(F_T(\zeta_i)-\frac{T+1}{2T}\right)=\frac{1}{T^{3/2}}\sum_{i=1}^{[Tu]}\sum_{j=1}^{T}\left(\mathbf{1}_{\{\zeta_j\le\zeta_i\}}-\frac{T+1}{2T}\right)$$
$$=\frac{1}{T^{3/2}}\sum_{i=1}^{[Tu]}\left\{\sum_{j=1}^{T}\left(\mathbf{1}_{\{\zeta_j\le\zeta_i\}}-\frac{1}{2}\right)\right\}-\frac{[Tu]}{2T^{3/2}}$$
$$=\frac{1}{T^{3/2}}\sum_{i=1}^{[Tu]}\left\{\sum_{j=1}^{[Tu]}\left(\mathbf{1}_{\{\zeta_j\le\zeta_i\}}-\frac{1}{2}\right)\right\}+\frac{1}{T^{3/2}}\sum_{i=1}^{[Tu]}\left\{\sum_{j=[Tu]+1}^{T}\left(\mathbf{1}_{\{\zeta_j\le\zeta_i\}}-\frac{1}{2}\right)\right\}-\frac{[Tu]}{2T^{3/2}}\,. \quad (17)$$

Let $f(x,y)=\mathbf{1}_{\{x\le y\}}-1/2$. Since $f(y,x)=-f(x,y)$, when $x\neq y$, we get that the first term in the r.h.s of (17) is equal to

$$\frac{1}{T^{3/2}}\sum_{i=1}^{[Tu]}\left\{\sum_{j=1}^{[Tu]}\left(\mathbf{1}_{\{\zeta_j\le\zeta_i\}}-\frac{1}{2}\right)\right\}=\frac{1}{T^{3/2}}\sum_{i=1}^{[Tu]}\left(\mathbf{1}_{\{\zeta_i\le\zeta_i\}}-\frac{1}{2}\right)=\frac{[Tu]}{2T^{3/2}}\,. \quad (18)$$

Thus,

$$\frac{1}{\sqrt{T}}\sum_{i=1}^{[Tu]}\left(F_T(\zeta_i)-\frac{T+1}{2T}\right)=\frac{1}{T^{3/2}}\sum_{i=1}^{[Tu]}\left\{\sum_{j=[Tu]+1}^{T}\left(\mathbf{1}_{\{\zeta_j\le\zeta_i\}}-\frac{1}{2}\right)\right\}\,. \quad (19)$$

Let $h(x,y)=\mathbf{1}_{\{x\le y\}}$. By using the Hoeffding's decomposition, we get that

$$\frac{1}{T^{3/2}}\sum_{i=1}^{[Tu]}\sum_{j=[Tu]+1}^{T}\left(h(\zeta_j,\zeta_i)-\frac{1}{2}\right)=\frac{1}{T^{3/2}}\sum_{i=1}^{[Tu]}\sum_{j=[Tu]+1}^{T}\left(F(\zeta_i)-\frac{1}{2}\right)$$
$$-\frac{1}{T^{3/2}}\sum_{i=1}^{[Tu]}\sum_{j=[Tu]+1}^{T}\left(F(\zeta_j)-\frac{1}{2}\right)+\frac{1}{T^{3/2}}\sum_{i=1}^{[Tu]}\sum_{j=[Tu]+1}^{T}\left(h(\zeta_j,\zeta_i)-F(\zeta_i)+F(\zeta_j)-\frac{1}{2}\right)\,, \quad (20)$$

F denoting the c.d.f of (ζ_t). Observe that since ε_t has a continuous c.d.f it is also the case of the c.d.f of ζ_t.

Let us now study the behavior of the first two terms in the r.h.s of (20):

$$\frac{T-[Tu]}{T^{3/2}}\sum_{i=1}^{[Tu]}\left(F(\zeta_i)-\frac{1}{2}\right)-\frac{[Tu]}{T^{3/2}}\sum_{j=[Tu]+1}^{T}\left(F(\zeta_j)-\frac{1}{2}\right)=X_{T,u}+Y_{T,u}\,.$$

Observe first that (ζ_i) is a strictly stationary process since it is defined as the sum of two strictly stationary processes. It is also a 1-dependent process in the sense of Example 1 p. 167 of Billingsley (1968) and $(F(\zeta_i))$ has the same properties. By using Theorem 20.1 of Billingsley

(1968), we get that, as T tends to infinity, $\{X_{T,u}\} \stackrel{d}{\Longrightarrow} \{(1-u)W(u)\}$, and that $\{Y_{T,u}\} \stackrel{d}{\Longrightarrow} \{u(W(1)-W(u))\}$, where $\{W(u)\}$ denotes the Wiener process. Since the process $(X_{T,u}, Y_{T,u})$ is tight and since for fixed u and s, $\mathrm{Cov}(X_{T,u}, Y_{T,s}) \to \mathrm{Cov}((1-u)W(u), s(W(1)-W(s)))$, as $T \to \infty$, $\{(X_{T,u}, Y_{T,u})\} \stackrel{d}{\Longrightarrow} \{((1-u)W(u), u(W(1)-W(u))\}$.

We deduce from this and Lemma 3 that

$$\frac{1}{T^{3/2}} \sum_{i=1}^{[Tu]} \sum_{j=[Tu]+1}^{T} \left(h(\zeta_j, \zeta_i) - \frac{1}{2} \right) \stackrel{d}{\longrightarrow} (1-u)W(u) - u(W(1)-W(u)) = W(u) - uW(1) \ , \quad (21)$$

which with (19) gives (16).

Let $w_i = F_T(\zeta_i) - (T+1)/(2T)$. Using similar arguments as those used in (Hamilton, 1994, p. 483)

$$\frac{1}{T} \sum_{t=2}^{T} \left\{ \frac{1}{\sqrt{T}} \sum_{j=1}^{t-1} \left(F_T(\zeta_j) - \frac{T+1}{2T} \right) \right\}^2$$
$$= \frac{1}{T} \left[\left(\frac{w_1}{\sqrt{T}} \right)^2 + \left(\frac{w_1 + w_2}{\sqrt{T}} \right)^2 + \cdots + \left(\frac{w_1 + \cdots + w_{T-1}}{\sqrt{T}} \right)^2 \right]$$
$$= \int_0^1 \left(\frac{1}{\sqrt{T}} \sum_{i=1}^{[Tu]} w_i \right)^2 \mathrm{d}t \stackrel{d}{\longrightarrow} \int_0^1 B(u)^2 \mathrm{d}u \ ,$$

by the continuous mapping theorem and (16), which concludes the proof of (11).

Lemma 3. *Under the assumptions of Theorem 2,*

$$\sup_{u \in [0,1]} \left| \frac{1}{T^{3/2}} \sum_{i=1}^{[Tu]} \sum_{j=[Tu]+1}^{T} \left(\mathbf{1}_{\{\zeta_j \le \zeta_i\}} - F(\zeta_i) + F(\zeta_j) - \frac{1}{2} \right) \right| = \sup_{u \in [0,1]} |Z_T(u)| = o_p(1) \ , \ \textit{as } T \to \infty \ . \quad (22)$$

Proof of Lemma 3. Observe that

$$Z_T(u) = \frac{1}{T^{3/2}} \sum_{i=1}^{[Tu]} \sum_{j=1}^{T} \left(\mathbf{1}_{\{\zeta_j \le \zeta_i\}} - F(\zeta_i) + F(\zeta_j) - \frac{1}{2} \right) - \frac{[Tu]}{2T^{3/2}} \ .$$

Thus it is enough to prove (22) when $Z_T(u)$ is replaced by

$$R_T(u) = \frac{1}{T^{3/2}} \sum_{i=1}^{[Tu]} \sum_{j=1}^{T} \left(\mathbf{1}_{\{\zeta_j \le \zeta_i\}} - F(\zeta_i) + F(\zeta_j) - \frac{1}{2} \right) \ .$$

We want to apply Lemma 5.2 of Borovkova et al. (2001) to $\{R_T(u),\ t \in [0,1]\}$. Let us first prove that for s and u such that $0 \le s \le u < s + \delta \le 1$,

$$R_T(u) - R_T(s) \le |R_T(s+\delta) - R_T(s)| + |W_T(s+\delta) - W_T(s)| + 2\frac{[T(s+\delta)] - [Ts]}{\sqrt{T}} \ ,$$

where

$$W_T(u) = \frac{T - [Tu]}{T^{3/2}} \sum_{i=1}^{[Tu]} \left(F(\zeta_i) - \frac{1}{2} \right) - \frac{[Tu]}{T^{3/2}} \sum_{j=[Tu]+1}^{T} \left(F(\zeta_j) - \frac{1}{2} \right) . \tag{23}$$

Observe that

$$R_T(u) - R_T(s) = \frac{1}{T^{3/2}} \sum_{i=[Ts]+1}^{[Tu]} \sum_{j=1}^{T} \left(\mathbf{1}_{\{\zeta_j \le \zeta_i\}} - F(\zeta_i) + F(\zeta_j) - \frac{1}{2} \right) . \tag{24}$$

Thus,

$$\begin{aligned}(R_T(u) - R_T(s)) - (R_T(s+\delta) - R_T(s)) &= -\frac{1}{T^{3/2}} \sum_{i=[Tu]+1}^{[T(s+\delta)]} \sum_{j=1}^{T} \left(\mathbf{1}_{\{\zeta_j \le \zeta_i\}} - F(\zeta_i) + F(\zeta_j) - \frac{1}{2} \right) \\ &= -\frac{1}{T^{3/2}} \sum_{i=[Tu]+1}^{[T(s+\delta)]} \sum_{j=1}^{T} \left(\mathbf{1}_{\{\zeta_j \le \zeta_i\}} - \frac{1}{2} \right) \\ &\quad + \frac{T}{T^{3/2}} \sum_{i=[Tu]+1}^{[T(s+\delta)]} \left(F(\zeta_i) - \frac{1}{2} \right) + \frac{[Tu] - [T(s+\delta)]}{T^{3/2}} \sum_{j=1}^{T} \left(F(\zeta_j) - \frac{1}{2} \right) .\end{aligned}$$

Using (23),

$$W_T(s+\delta) - W_T(s) = \frac{1}{\sqrt{T}} \sum_{i=[Ts]+1}^{[T(s+\delta)]} \left(F(\zeta_i) - \frac{1}{2} \right) + \frac{[Ts] - [T(s+\delta)]}{T^{3/2}} \sum_{j=1}^{T} \left(F(\zeta_j) - \frac{1}{2} \right) ,$$

and thus, we get that

$$\begin{aligned}&(R_T(u) - R_T(s)) - (R_T(s+\delta) - R_T(s)) - (W_T(s+\delta) - W_T(s)) \\ &\quad = -\frac{1}{T^{3/2}} \sum_{i=[Tu]+1}^{[T(s+\delta)]} \sum_{j=1}^{T} \left(\mathbf{1}_{\{\zeta_j \le \zeta_i\}} - \frac{1}{2} \right) - \frac{1}{\sqrt{T}} \sum_{i=[Ts]+1}^{[Tu]} \left(F(\zeta_i) - \frac{1}{2} \right) \\ &\qquad + \frac{[Tu] - [Ts]}{T^{3/2}} \sum_{j=1}^{T} \left(F(\zeta_j) - \frac{1}{2} \right) \le \frac{3}{2} \frac{[T(s+\delta)] - [Ts]}{\sqrt{T}} .\end{aligned}$$

Since we can follow the same line of reasoning for $R_T(s) - R_T(u)$, we get Condition (5.10) of Lemma 5.2 in Borovkova et al. (2001) with $\alpha = 1/2$.

Let us now check Condition (5.9) of Lemma 5.2 in Borovkova et al. (2001). Observe that for $0 \le s \le u \le 1$,

$$\begin{aligned}W_T(u) - W_T(s) &= \frac{T - ([Tu] - [Ts])}{T^{3/2}} \sum_{i=[Ts]+1}^{[Tu]} \left(F(\zeta_i) - \frac{1}{2} \right) \\ &\quad + \frac{[Ts] - [Tu]}{T^{3/2}} \left\{ \sum_{i=1}^{[Ts]} \left(F(\zeta_i) - \frac{1}{2} \right) + \sum_{i=[Tu]+1}^{T} \left(F(\zeta_i) - \frac{1}{2} \right) \right\} .\end{aligned}$$

Thus, for some positive constant C_1

$$\mathbb{E}[(W_T(u)-W_T(s))^4] \leq C_1 \frac{(T-([Tu]-[Ts]))^4}{T^6} \mathbb{E}\left[\left\{\sum_{i=[Ts]+1}^{[Tu]}\left(F(\zeta_i)-\frac{1}{2}\right)\right\}^4\right]$$
$$+ C_1 \frac{([Tu]-[Ts])^4}{T^6} \mathbb{E}\left[\left\{\sum_{i=1}^{[Ts]}\left(F(\zeta_i)-\frac{1}{2}\right)\right\}^4\right]$$
$$+ C_1 \frac{([Tu]-[Ts])^4}{T^6} \mathbb{E}\left[\left\{\sum_{i=[Tu]+1}^{T}\left(F(\zeta_i)-\frac{1}{2}\right)\right\}^4\right] .$$

Using that (ζ_i) is a 1-dependent process and that $\mathbb{E}[(F(\zeta_i)-1/2)]=0$, $|F(\zeta_i)-1/2| \leq 1/2$ for all $i \geq 1$, there exist positive constants C_2 and C_3 such that

$$\mathbb{E}[(W_T(u)-W_T(s))^4] \leq C_2 \frac{(T-([Tu]-[Ts]))^4}{T^6} \left\{([Tu]-[Ts])+([Tu]-[Ts])^2\right\}$$
$$+ C_2 \frac{([Tu]-[Ts])^4}{T^6} \left\{[Ts]+[Ts]^2+(T-[Tu])+(T-[Tu])^2\right\}$$
$$\leq C_3 \left\{\frac{([Tu]-[Ts])^2}{T^2}+\frac{([Tu]-[Ts])}{T^2}\right\} ,$$

which is Condition (5.10) of Lemma 5.2 in Borovkova et al. (2001) with $h=1$, $g(t)=t$ and $r=4$.

Let us now check Condition (5.8) of Lemma 5.2 in Borovkova et al. (2001). Using (24), we get that for $0 \leq s \leq u \leq 1$,

$$\mathbb{E}[(R_T(u)-R_T(s))^2] = \mathbb{E}\left[\left\{\frac{1}{T^{3/2}}\sum_{i=[Ts]+1}^{[Tu]}\sum_{j=1}^{[Ts]} A_{i,j} + \frac{1}{T^{3/2}}\sum_{i=[Ts]+1}^{[Tu]}\sum_{j=[Tu]+1}^{T} A_{i,j} + \frac{[Tu]-[Ts]}{2T^{3/2}}\right\}^2\right]$$
$$\leq \frac{3}{T^3}\mathbb{E}\left[\left\{\sum_{i=[Ts]+1}^{[Tu]}\sum_{j=1}^{[Ts]} A_{i,j}\right\}^2\right] + \frac{3}{T^3}\mathbb{E}\left[\left\{\sum_{i=[Ts]+1}^{[Tu]}\sum_{j=[Tu]+1}^{T} A_{i,j}\right\}^2\right] + \frac{3([Tu]-[Ts])^2}{2T^3} ,$$

where $A_{i,j} = \mathbf{1}_{\{\zeta_j \leq \zeta_i\}} - F(\zeta_i) + F(\zeta_j) - 1/2$. Using that (ζ_i) is a 1-dependent process, that $\mathbb{E}(A_{i,j}) = 0$ for all $i \neq j$ and that $\mathbb{E}(A_{i,j}A_{k,\ell}) = 0$ as soon as the distances between the different indices i,j,k,ℓ is larger than 1, there exist positive constants C_4, C_5 and C_6 such that

$$\mathbb{E}[(R_T(u)-R_T(s))^2] \leq C_4 \frac{([Tu]-[Ts])[Ts]}{T^3} + C_4 \frac{([Tu]-[Ts])(T-[Tu])}{T^3} + C_5 \frac{([Tu]-[Ts])^2}{T^3}$$
$$\leq C_6 \frac{|u-s|}{T} \leq C_6 \frac{|u-s|^{1-\nu}}{T} ,$$

for some positive ν which gives Condition (5.8) of Lemma 5.2 in Borovkova et al. (2001) with $\beta=1$ and $\gamma=1-\nu$ and thus concludes the proof of Lemma 3. □

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